QUANTUM CATASTROPHES, BLACK HOLES, & RAINBOWS

SUMMARY: Caustics are a well-known phenomena in optics where light is focused due to naturally occurring lensing effects. Some examples include rainbows, the wavy lines at the bottom of swimming pools, and the cusp-shaped focusing of light that can be seen inside a coffee mug. By studying a flowing, ultracold quantum gas, we were able to show that the universal mathematics of caustics also locally describes the thermal radiation predicted to be emitted by black holes [1].

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Gravitational black holes (BHs) are at the forefront of where our current understanding of physics fails; our best theories of gravity and quantum mechanics do not play nicely near such cosmic entities. Despite the lack of a complete and successful theory of quantum gravity, it has been shown that BHs are expected to emit thermal Hawking radiation (HR) as if they were black bodies (like the glowing hot heating elements on a stove) [2]. However, at BH event horizons the HR appears to possess infinite energy, heralding a breakdown of the predicted physics [3]. To overcome the experimental difficulty in studying BHs and to help address this HR breakdown, laboratory analogues of BHs have been created in flowing ultracold gases [4]: if the flow speed of the gas exceeds the speed of sound, then soundwaves are unable to propagate against the flow and an analogue of an event horizon occurs. Remarkably, HR is produced near this sonic horizon in the form of oppositely travelling pairs of soundwaves. A sonic black hole is depicted in Figure 1.

The non-physical energy divergence of HR at the horizon is in fact an example of a quantum catastrophe [5]: a generalization of a well-known effect in optics, namely caustics [6]. Caustics are regions where light focuses due to naturally occurring lensing effects. Some examples are displayed in Figure 2, which include rainbows, the wavy lines at the bottom of swimming pools, and the cusp-shaped focusing of light that can be seen inside a coffee mug. In the geometrical ray approximation of optics, caustics are where a finite number of rays are focused into an infinitely small region, leading to an infinite ray density and thus a non-physical prediction of infinite intensity. This heralds a breakdown

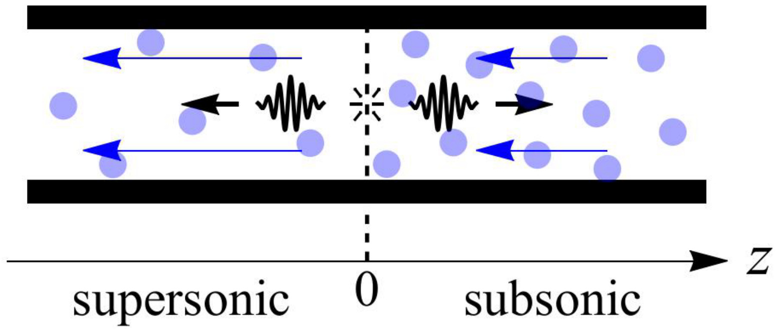

Figure 1. Schematic representation of a sonic BH formed by flowing an ultracold gas in-between two trapping walls (thick black lines), where z denotes the axial distance from the BH horizon. The blue circles are the constituent particles, and the blue arrows indicate their direction and magnitude of flow. The vertical dashed line at $z = 0$ indicates the position of a sonic event horizon so that the flow is supersonic when $z < 0$ (inside the BH), and subsonic when $z > 0$ (outside). The localized oscillations represent the spontaneously produced sonic HR which forms near and propagates away from the horizon: one particle escapes the BH and moves rightwards, while the other gets swept leftwards into the BH.

of the ray picture and motivates the need for a wave-based description, which the field of mathematics known as catastrophe theory [7] precisely provides. Catastrophe theory tells us how to cure the ray singularities of caustics and that each one falls into a hierarchy of families with certain universal shapes and properties. The two simplest cases are known as fold and cusp caustics and are respectively described by the following two integrals: the Airy (left) and the Pearcey (right) functions [8], where (x, y) are control parameters (e.g., position coordinates) characterizing the geometry of the caustics.

$$\mathrm{Ai}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\mathrm{k}^3/3+\mathrm{kx})}\,\mathrm{dk} \qquad\qquad \mathrm{Pe}(x,y) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\mathrm{k}^4/4+\mathrm{xk}^2/2+\mathrm{ky})}\,\mathrm{dk}$$

The frequency of the HR (ω) in our ultracold gas is approximated by the equation below [9], where u describes the sub- to super-sonic flow speed of the gas, k is the HR momentum (wavenumber), c the average speed of sound within the gas, and Λ a quantum length scale related to the interatomic distance between constituent particles (analogous to the existence of a Planck scale that might discretize spacetime in quantum gravity).

$$\omega - uk \approx ck + \Lambda k^3$$

The term proportional to Λ is relatively small and thus often ignored, resulting in the analogous relativistic form: $\omega - uk \approx ck$. However, the HR breakdown at the sonic horizon can in fact be resolved by retaining the quantum length scale term, and by doing so one finds that the HR is

Figure 2. Examples of caustics due to the refraction and reflection of sunlight. **Left:** Each band of colour in a rainbow is formed due to two waves focusing together and is mathematically described by an Airy function. **Middle:** The wavy lines at the bottom of swimming pools are due to the focusing of waves by the noisy surface of the water. **Right:** The cusp shaped focusing in the bottom of a coffee mug is mathematically described by a Pearcey function, where 3 waves focus together thanks to the cylindrical shape of the mug.

approximately described by the integral below [9]. We denote this the 'log-Airy' integral because it appears similar to the Airy function, but is modified by a logarithmic term within the integrand's phase.

$$\mathrm{LAi}(z, \omega) = \int_{-\infty}^{\infty} \frac{1}{k} \mathrm{e}^{\mathrm{i}\left(\mathrm{k}^3/3+\mathrm{kx}+\omega \ln(\mathrm{k})\right)}\, \mathrm{dk}$$

To understand how the HR forms and behaves near the horizon, we applied a modified version of the method of steepest descents [10] — a technique used to approximate integrals. For the log-Airy this involves deforming the 1D integration path from the real k-axis to new 3D contours that traverse complex k-space. The contours must pass through saddlepoints of the phase whilst simultaneously starting and ending in specific regions. Doing so ensures that each relevant contour-saddle pair physically represents a Hawking particle during a particular segment of its evolution. Combining every contour-saddle contribution thus yields a complete description of the HR dynamics near the horizon. It turns out that for the log-Airy integral the physical choice of contours is non-trivial. Previous approaches introduced additional approximations [9] which, although simplified the choice of steepest descent contours, led to a breakdown of the near-horizon physics, defeating the original purpose of keeping the quantum length scale term.

To overcome the challenge of non-trivial contour selection and the need for further approximation, we applied a simple exponential coordinate transformation, $k \to e^w$. This unfolds the infinitely spiraling structure of the 3D complex k-space into a flat, infinitely extended 2D complex w-plane [11], providing a more intuitive framework for identifying which contour-saddle contributions are relevant for describing the HR, as shown in Figure 3. Through our unfolded steepest descent analysis, we discovered that the log-Airy exhibits properties of the Airy function while also displaying steepest descent behavior locally equivalent to that of the Pearcey function [1]. As a result, the universal

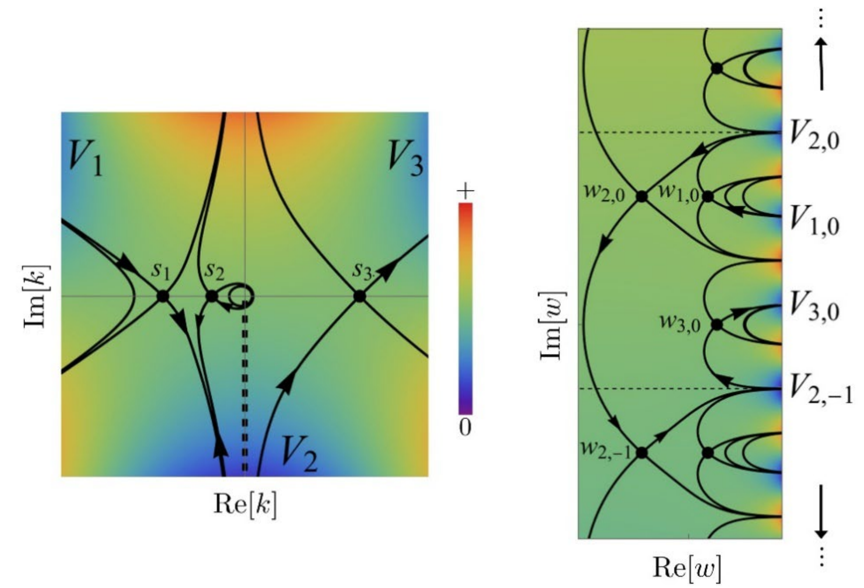

Figure 3. **Left:** 2D steepest descent diagram before unfolding, obtained by projecting the spiraling complex 3D contours (black solid lines) into 2D from $\theta = 0 \rightarrow 2\pi$ (starting at the black dashed line: a branch cut) for a particular choice of parameters (z, ω). To physically describe the HR, the contours must start and terminate in the various $V_{1,2,3}$ regions and simultaneously pass through saddlepoints (black dots)**.** Due to the presence of the branch cut the physical contributions are non-trivial in k-space. **Right:** Unfolded steepest descent diagram. In this w-space the correct contour-saddle pairs, and therefore the correct physical interpretation of the HR, becomes much more apparent.

mathematics of caustics — catastrophe theory — also applies to the log-Airy integral. This caustic-motivated approach uses the same universal physics that describes rainbows, offering a more accurate description of sonic HR by retaining the near-horizon behavior! It also bridges the concept of quantum catastrophes to the rigorous mathematics of catastrophe theory, reinforcing the idea that analogue models of gravity are valuable tools for investigating quantum-gravitational phenomena due to their universality.

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