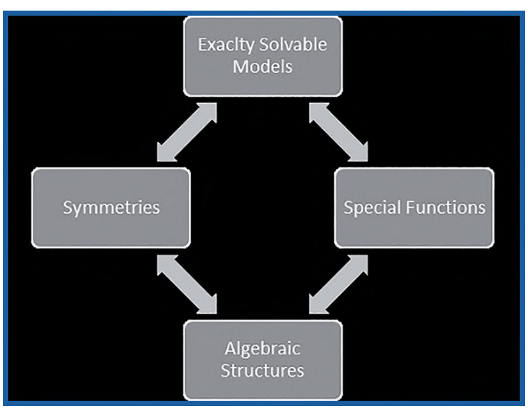
In 2016, the Division of Theoretical Physics (DTP), in partnership with the Winnipeg Institute for Theoretical Physics (WITP), created a PhD Thesis Prize competition for the best thesis in theoretical physics by any student receiving their PhD degree from a Canadian university in the current or prior calendar year (see http://www.cap.ca/en/div/dtp/thesis-prize). The DTP is pleased to announce that the recipient of the 2015-16 DTP-WITP Thesis Prize is Vincent X. Genest. Dr. Genest was awarded his PhD by the Université de Montréal in 2015 for the work "Algebraic Structures, Superintegrable Systems and Orthogonal Polynomials". A summary of Dr. Genest's thesis work appears below.

ALGEBRAIC STRUCTURES, SUPERINTEGRABLE SYSTEMS AND ORTHOGONAL POLYNOMIALS

BY VINCENT X. GENEST

The search for exactly solvable systems and their analysis has a long-standing tradition in mathematical physics. This is so because the study of exactly solvable models provides significant insight into fundamental principles, but also because these models form a bedrock for the study of symmetries. Indeed, exact solvability is often made possible by the presence of symmetries, which can be described by algebraic structures (e.g. Lie groups). In many cases, observables in exactly solvable models are expressed in terms of special functions (e.g. orthogonal polynomials), whose analytic properties encode the symmetries of the system. The interaction between these elements can be presented schematically as follows:



SUMMARY

This article gives a peek at some of the results of the thesis "Algebraic Structures, Superintegrable Systems, and Orthogonal polynomials". The thesis won the 2016 CAP-DTP/WITP Thesis Prize, as well as the 2016 Doctoral Prize from the Canadian Mathematical Society.

Unravelling new exactly solvable and superintegrable models, new symmetries, new algebraic structures and their representations, as well as new families of orthogonal polynomials, and exploring their import to all the other fields of the above diagram is the leitmotiv of my thesis. The thesis is comprised of 28 articles written in collaboration with other physicists and mathematicians, and is concerned with the following topics:

1. The algebraic characterization of families of multivariate orthogonal functions and their applications to physics.
2. The study of the Bannai-Ito algebra and its relation with the Bannai-Ito hierarchy of univariate orthogonal polynomials, as well as its relation to novel families of exactly solvable and superintegrable quantum systems.
3. The determination of the link between quantum superintegrable systems with 2^{nd} order constants of motion and the recoupling of Lie algebra and superalgebra representations.
4. The study of the symmetries and algebraic structures associated to 2D and 3D Dunkl-type systems, which are governed by Hamiltonians involving reflection operators.
5. The algebraic characterization of matrix multi-orthogonal polynomials and their applications to squeezed-coherent states for finite oscillator models.

In the interest of brevity, I shall here touch upon a small subset of the novel results obtained in regards to the first two topics.

MULTIVARIATE ORTHOGONAL POLYNOMIALS AND APPLICATIONS

In a series of papers [1–3], we considered matrix elements of the unitary representations of the rotation, Lorentz and Euclidean groups on oscillator states. We showed that these matrix elements have the general form

$$< n_1, n_2, \ldots, n_d | U(g) | x_1, x_2, \ldots, x_d > \\ = \sqrt{\omega(x_1, x_2, \ldots, x_d)} P^{(g)}_{n_1, n_2, \ldots, n_d}(x_1, x_2, \ldots, x_d),$$

Vincent X. Genest, <vxgenest@mit.edu>, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139 USA

where $P^{(g)}_{n_1,n_2,\ldots,n_d}(x_1,x_2,\ldots,x_d)$ are orthogonal polynomials in the discrete variables $x_i \in \{1, 2, \ldots\}$, where $\omega(x_1, x_2, \ldots, x_d)$ is their orthogonality weight, and where $U(g)$ stands for the unitary representation of the group element g. For the rotation and Lorentz groups, the polynomials were respectively identified as the multivariate Krawtchouk and Meixner polynomials. This physical interpretation involving the oscillator states allowed to completely characterize these polynomials, to derive new properties, and to cast them in a clear physical framework. In the case of the Euclidean group, the analysis led to the discovery and characterization of a new family of multivariate Charlier polynomials. As for direct physical applications, the 2-variable Meixner et Charlier polynomials were seen to arise as the exact solutions in discrete oscillator models that share the same *SU(2)* symmetry as their continuous analog; see for example [4]. The 2-variable Krawtchouk polynomials played a central role in the design of a two-dimensional spin lattice that exhibits a kind of perfect state transfer [5].

A DUNKL-DIRAC EQUATION AND THE BANNAI-ITO ALGEBRA

In [6], we considered the null-solutions of the so-called Dirac-Dunkl operator in $\mathbb{R}^3$ associated to the reflection group $\mathbb{Z}_2^3$. These null-solutions ψ satisfy the equation $\underline{D}\psi = 0$, where $\underline{D}$ is the Dirac-Dunkl operator defined as

$$\underline{D} = \sigma_1 T_1 + \sigma_2 T_2 + \sigma_3 T_3.$$

In the above, σ_i are the Pauli matrices and T_i are the Dunkl operators

$$T_i = \partial_i + \frac{\mu_i}{x_i}(1 - r_i),$$

where ∂_i is the derivative with respect to the coordinate x_i, where μ_1, μ_2, μ_3 are positive parameters and where r_i is the reflection operator in x_i, e.g. $r_1 f(x_1, x_2, x_3) = f(-x_1, x_2, x_3)$. As a result, this operator can be viewed as a three-parameter deformation of the Dirac operator in $\mathbb{R}^3$; its spherical component can similarly be interpreted as a three-parameter deformation of the spin-orbit Hamiltonian. We constructed a complete set of constants of motion that commute with $\underline{D}$ and showed that these symmetries, denoted by K_1, K_2, K_3 (with $[K_i, \underline{D}] = 0$ for $i = 1, 2, 3$) satisfy the Bannai-Ito algebra

$$\{K_1, K_2\} = K_3 + \delta_3, \{K_2, K_3\} = K_1 + \delta_1, \{K_3, K_1\} = K_3 + \delta_3,$$

where $\{A, B\}$ stands for the anticommutator, and where δ_1, δ_2, δ_3 are central elements (which are constants on the null-solutions ψ). We constructed an explicit basis for the null-solutions ψ involving Jacobi polynomials, and showed that these solutions transform irreducibly under the action of the Bannai-Ito algebra.

The Bannai-Ito algebra is one example of the novel algebraic structures that was uncovered in my thesis and related to several exactly solvable and superintegrable systems. As an example, this algebra was also seen to arise as the invariance algebra for a system three para-Bose oscillators [7].

CONCLUSION

In my thesis, I explored the deep interplay between exact solvability, symmetries, algebraic structures and their representations, and special functions, in particular orthogonal polynomials of one or many variables. There is no doubt to me that there is still much to explore in that area. Furthermore, one can expect that the novel algebraic structure exhibited in my thesis, in light of their naturalness, will emerge in other applications to mathematics or physics.

ACKNOWLEDGEMENTS

The author would like to acknowledge the financial support of the Natural Science and Engineering Research Council of Canada. The author would like to thank Luc Vinet for his outstanding supervision, as well as all of his co-authors.

REFERENCES

1. V.X. Genest, L. Vinet, and A. Zhedanov, “The multivariate Krawtchouk polynomials as matrix elements of the rotation group representations on oscillator states”, *J. Phys. A*, **46**, 505203 (2013).
2. V.X. Genest, H. Miki, L. Vinet, and A. Zhedanov, “The multivariate Meixner polynomials as matrix elements of SO(d,1) representations on oscillator states”, *J. Phys. A*, **47**, 045207 (2014).
3. V.X. Genest, H. Miki, L. Vinet, and A. Zhedanov, “The multivariate Charlier polynomials as matrix elements of the Euclidean group representations on oscillator states”, *J. Phys. A*, **47**, 215204 (2014).
4. J. Gaboriaud, V.X. Genest, J. Lemieux, and L. Vinet, “A superintegrable discrete oscillator and two-variable Meixner polynomials”, *J. Phys. A*, **48**, 415202 (2015).
5. V.X. Genest, H. Miki, L. Vinet, and A. Zhedanov, “Spin lattices, state transfer and bivariate Krawtchouk polynomials”, *Can. J. Phys.*, **93**, 979-984 (2014).
6. H. De Bie, V.X. Genest, and L. Vinet, “A Dirac-Dunkl equation on S^2 and the Bannai-Ito algebra”, *Commun. Math. Phys.*, **344**, 447-464 (2016).
7. V.X. Genest, L. Vinet, and A. Zhedanov, “The Bannai-Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra”, *Proc. Amer. Math. Soc.*, **142**, 1545-1560 (2014).